Nonlinear quantum regime of the x-ray Compton laser

H. K. Avetissian* and G. F. Mkrtchian
Department of Theoretical Physics, Plasma Physics Laboratory, Yerevan State University, 1 A. Manukian, Yerevan 375049, Armenia

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In this work a scheme of x-ray coherent radiation generation in the nonlinear quantum regime by means of mildly relativistic high density electron beams and a strong pump laser field is investigated. The consideration is based on a self-consistent set of Maxwell and relativistic quantum kinetic equations. The coupled equations are solved in the slowly varying envelope approximation.

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I. INTRODUCTION

The problem of creation of short-wave coherent radiation sources in its general aspects reduces to the implementation of free electron lasers (FEL's) [1]. The main advantage of a FEL lies in the fact that the emission frequency ω' is continuously Doppler upshifted by several orders of magnitude ($\omega' \sim \gamma_L^2 \omega$, γ_L being the Lorentz factor) with respect to the frequency of the pump field ω. In particular, various schemes of x-ray FEL's have been considered based on the coherent accumulation of ultrarelativistic electron beam radiation in the undulator and on Compton backscattering, channeling, transition and diffraction radiation [2]. Among these versions at present the undulator scheme [3] is being actively developed. Although the amplifying frequencies are still far from x-ray frequencies the main hopes for an efficient x-ray FEL remain connected with the undulator scheme. For this purpose two international projects TESLA and LCLS [4] are currently being implemented. However, the other versions mentioned such as Compton backscattering [5] may appear more reasonable in practice for x-ray FEL's due to easier setup requirements; in particular, the use of electron beams of considerably lower energies.

In contrast with conventional laser devices in atomic systems, the FEL is usually reckoned as a classical device (exhibiting also non-Poissonian photon statistics [6]). But this is not a universal property of FEL's as in some cases quantum effects may play a significant role. In the quantum description [7] the small-signal gain of the FEL is usually represented as a convolution integral of the electron beam momentum distribution with the difference between the probability distributions of emission and absorption per photon. Since the electron recoils in opposite directions depending on whether it emits or absorbs photons with the same wave vector $\mathbf{k}'$ the resonant momenta of an electron for emission p_e and absorption p_a are different. Hence, the probability distributions of emission and absorption are centered at p_e and p_a, and when these distributions are much narrower than the spread of the electron beam distributions $f(p)$, the small-signal gain is proportional to the so called "population inversion" $f(p_e)-f(p_a)$. In the quasiclassical limit when photon energy $\hbar\omega'$ satisfies the condition

$$\hbar\omega' \ll \max\{\Delta\varepsilon_\gamma, \Delta\varepsilon_\vartheta, \Delta\varepsilon_L\} \tag{1.1}$$

($\Delta\varepsilon_\gamma$ and $\Delta\varepsilon_\vartheta$ are the resonance widths due to energetic and angular spreads, and $\Delta\varepsilon_L$ the resonance width caused by the finite interaction length) the quantum expression for the gain coincides with its classical counterpart, being antisymmetric about the classical resonant momentum $p_c=(p_e+p_a)/2$ and proportional to the derivative of the momentum distribution $df(p)/dp$ at p_c. The result is that amplification takes place only if the initial momentum distribution is centered above p_c as the electrons whose momenta are above p_c contribute on average to the small-signal gain, and the electrons whose momenta are below p_c contribute on average to the corresponding loss. This severely limits the FEL gain performance at short wavelengths [1]. In the more conventional undulator devices, to achieve the x-ray frequency domain one should increase the electron energies up to several GeV, which in turn significantly reduces the small-signal gain ($\sim \gamma_L^{-3}$). To achieve the x-ray domain with moderate relativistic electron beams (energy of electrons $\lesssim 50$ MeV), the frequency of electron self-oscillation should be high enough $\sim 10^{14}$–10^{15} s^{-1} (in the undulator 10^{10} s^{-1}). The latter can be realized, e.g., in the Compton backscattering scheme suggested over 30 years ago [5].

Another way to increase the efficiency of a FEL is to achieve the quantum regime of generation

$$\hbar\omega' \gtrsim \max\{\Delta\varepsilon_\gamma, \Delta\varepsilon_\vartheta, \Delta\varepsilon_L\} \tag{1.2}$$

as in this case the absorption and emission line shapes are separated and the simultaneous absorption of a probe wave is excluded. From this point of view the scheme of an x-ray Compton laser has an advantage with respect to the conventional undulator devices connected with the satisfaction of condition (1.2) for the quantum regime of generation. To achieve this condition for current FEL devices operating in undulators is problematic as it presumes severe restrictions on the beam spread [8]. So the scheme of an x-ray Compton laser in the quantum regime of generation is preferable, since it requires considerably lower energies of the electron beam and moderate restrictions on the beam spreads.

In this work, a scheme of x-ray coherent radiation generation in the nonlinear quantum regime by means of a mildly relativistic high density electron beam and a strong pump laser field is investigated. This makes it possible to achieve the quantum regime of generation at x-ray frequencies as

*Electronic address: avetissian@ysu.am

well, due to radiation of high harmonics of Doppler-shifted pump frequencies in the strong laser field. The latter effect is already used in high power laser technology [9]. Therefore, the consideration of nonlinear electron-laser interaction schemes in induced free-free transitions is realistic In addition, concerning the further process of x-ray radiation amplification it is necessary to realize a single-pass FEL, as long as the construction of resonators in the x-ray domain is problematic. In the linear regime this demands very long interaction, lengths. So the investigation of nonlinear regimes of x-ray radiation generation still remains a topical problem. In this work the main emphasis is on the nonlinear regime of generation. The consideration is based on a self-consistent set of Maxwell and quantum kinetic equations. Because the energy-momentum levels are not equidistant, the probe wave resonantly couples only two Volkov states, and the coupled equations are solved in the slowly varying envelope approximation.

This paper is organized as follows. In Sec. II we obtain the self-consistent set of equations arising from the second quantization formalism. The steady-state regimes of amplification for x-ray generation are considered in Sec. III. Finally, a discussion of the results obtained and the conclusions are given in Sec. IV.

II. SELF-CONSISTENT SET OF MAXWELL AND RELATIVISTIC QUANTUM KINETIC EQUATIONS

As is known the Dirac equation allows an exact solution in the field of a plane electromagnetic (em) wave (Volkov solution) [10,11]. Although the Volkov states are not stationary, as there are no real transitions in the monochromatic em wave (due to violation of energy and momentum conservation laws) the state of a particle in an em wave can be characterized by the quasimomentum $\mathbf{q}$ and polarization σ [10] (the particle and antiparticle solutions are also separated). We will consider a given pump em wave that is described by the four-vector potential

$$A^{\mu}=(0,\mathbf{A}), \tag{2.1}$$

where

$$\mathbf{A}=(A_0\cos kx, gA_0\sin kx, 0), \tag{2.2}$$

$x=(ct,\mathbf{r})$ is the four-component radius vector (c is the light speed in vacuum), and

$$k\equiv\left(\frac{\omega}{c},\mathbf{k}\right) \tag{2.3}$$

is the four-wave-vector. Here and in what follows for the four-component vectors we have chosen the metric $a\equiv a^{\mu}=(a_0,\mathbf{a})$ and ax is the relativistic scalar product

$$ax\equiv a^{\mu}x_{\mu}=a_0x_0-\mathbf{a}\cdot\mathbf{x}.$$

In Eq. (2.2) $g=\pm 1$ correspond to a circularly polarized electromagnetic wave (cw), while $g=0$ corresponds to a linearly polarized one (lw). The particle state $|\mathbf{q},\sigma\rangle$ in the field (2.2) can be presented in the form [11]

$$|\mathbf{q},\sigma\rangle=\left[1+\frac{e\hat{k}\hat{A}}{2c(kp)}\right]\frac{u_{\sigma}(p)}{\sqrt{2q_0V}}\exp\left[-\frac{i}{\hbar}\left\{qx-\frac{eA_0}{c(pk)}(p_x\sin kx-gp_y\cos kx)-\frac{e^2A_0^2}{8c^2(pk)}(1-g^2)\sin(2kx)\right\}\right]. \tag{2.4}$$

Here we have introduced the notation $\hat{a}=a^{\mu}\gamma_{\mu}$, where $\gamma^{\mu}=(\gamma_0,\gamma)$ are Dirac matrices. $q=(q_0/c,\mathbf{q})$ is the average four-kinetic-momentum or "quasimomentum" of the electron in the em wave (emw) field, which is defined via the free electron four-momentum $p=(\varepsilon_0/c,\mathbf{p})$ and the relativistic invariant parameter of the wave intensity ξ by the following equation:

$$q=p+k\frac{m^2c^2}{4k\cdot p}(1+g^2)\xi^2, \qquad \xi=\frac{eA_0}{mc^2},$$

where m and e are the particle mass and charge, respectively. In Eq. (2.4) $u_{\sigma}(p)$ is the bispinor amplitude of a free Dirac particle with polarization σ, V is the volume of the periodicity box (in what follows we will put $V=1$), and it is assumed that

$$\overline{u}u=2mc^3,$$

where $\overline{u}=u^{\dagger}\gamma_0$; $u^{\dagger}$ denotes the transposition and complex conjugation of u. So the states (2.4) are normalized by the condition

$$\langle\mathbf{q}',\sigma'\|\mathbf{q},\sigma\rangle=\delta_{\mathbf{q},\mathbf{q}'}\delta_{\sigma,\sigma'},$$

where $\delta_{\mu\mu'}$ is the Kronecker symbol.

We assume the probe emw to be linearly polarized with the carrier frequency ω' and four-vector potential

$$A_w=\frac{e_1}{2}\{A_e(t,\mathbf{r})e^{-ik'x}+k\cdot c\}, \tag{2.5}$$

where $A_e(t,\mathbf{r})$ is a slowly varying envelope, $k'=(\omega'/c,\mathbf{k}')$ is the four-wave-vector, and e_1 is the unit polarization four-vector $e_1k'=0$.

Cast in the second quantization formalism, the Hamiltonian is

$$\hat{H}=\int\hat{\Psi}^{\dagger}\hat{H}_0\hat{\Psi}d\mathbf{r}+\hat{H}_{int}, \tag{2.6}$$

where $\hat{\Psi}$ is the fermionic field operator, $\hat{H}_0$ is the one-particle Hamiltonian in the plane EMW (2.1), and the interaction Hamiltonian is

$$\hat{H}_{int}=\frac{1}{c}\int\hat{j}A_wd\mathbf{r} \tag{2.7}$$

with the current density operator

$$\hat{j}=e\hat{\Psi}^{\dagger}\gamma_0\gamma\hat{\Psi}. \tag{2.8}$$

We pass to the furry representation and write the Heisenberg field operator of the electron in the form of an expansion in the quasistationary Volkov states (2.4)

$$\hat{\Psi}(\mathbf{x},t)=\sum_{\mathbf{q},\sigma}\hat{a}_{\mathbf{q},\sigma}(t)|\mathbf{q},\sigma\rangle, \tag{2.9}$$

where we have excluded the antiparticle operators, since contributions of particle-antiparticle intermediate states will lead only to small corrections to the processes considered. The creation and annihilation operators $\hat{a}^{\dagger}_{\mathbf{q},\sigma}(t)$ and $\hat{a}_{\mathbf{q},\sigma}(t)$ associated with positive energy solutions satisfy the anticommutation rules at equal times

$$\{\hat{a}^{\dagger}_{\mathbf{q},\sigma}(t),\hat{a}_{\mathbf{q}',\sigma'}(t')\}_{t=t'}=\delta_{\mathbf{q},\mathbf{q}'}\delta_{\sigma,\sigma'}, \tag{2.10}$$

$$\{\hat{a}^{\dagger}_{\mathbf{q},\sigma}(t),\hat{a}^{\dagger}_{\mathbf{q}',\sigma'}(t')\}_{t=t'}=\{\hat{a}_{\mathbf{q},\sigma}(t),\hat{a}_{\mathbf{q}',\sigma'}(t')\}_{t=t'}=0.$$

Taking into account Eqs. (2.9), (2.8), (2.7), and (2.4), the second quantized interaction Hamiltonian can be expressed in the form

$$\begin{aligned}\hat{H}_{int}=&\frac{e}{2c}\overline{A}_e^{*}\sum_s\sum_{\mathbf{q}_1,\sigma_3,\sigma_4}\hat{a}^{\dagger}_{\mathbf{q}_1-\hbar\mathbf{k}'+s\hbar\mathbf{k},\sigma_4}\hat{a}_{\mathbf{q}_1,\sigma_3}\\&\times\langle\mathbf{q}_1-\hbar\mathbf{k}'+s\hbar\mathbf{k},\sigma_4\|s\|\mathbf{q}_1,\sigma_3\rangle e^{i\Delta(\mathbf{q}_1-\hbar\mathbf{k}'+s\hbar\mathbf{k},\mathbf{q}_1)t}\\&+\frac{e}{2c}\overline{A}_e\sum_s\sum_{\mathbf{q}_1,\sigma_3,\sigma_4}\hat{a}^{\dagger}_{\mathbf{q}_1+\hbar\mathbf{k}'+s\hbar\mathbf{k},\sigma_4}\\&\times\hat{a}_{\mathbf{q}_1,\sigma_3}\langle\mathbf{q}_1+\hbar\mathbf{k}'+s\hbar\mathbf{k},\sigma_4\|s\|\mathbf{q}_1,\sigma_3\rangle\\&\times e^{-i\Delta(\mathbf{q}_1,\mathbf{q}_1+\hbar\mathbf{k}'+s\hbar\mathbf{k},)t}.\end{aligned} \tag{2.11}$$

Here

$$\begin{aligned}&\langle\mathbf{q}_2,\sigma_2\|s\|\mathbf{q}_1,\sigma_1\rangle\\&=\frac{\overline{u(p_2)}}{2\sqrt{q_{01}q_{02}}}\Bigg\{\left(\hat{e}_1-\frac{g^2e^2A_0^2(ke_1)\hat{k}}{2c^2(p_1k)(p_2k)}\right)\Lambda_0\\&\quad-eA_0\left(\frac{\gamma_x\hat{k}\hat{e}_1}{2c(p_1k)}+\frac{\hat{e}_1\hat{k}\gamma_x}{2c(p_2k)}\right)\Lambda_1\\&\quad-eA_0\left(\frac{\gamma_y\hat{k}\hat{e}_1}{2c(p_1k)}+\frac{\hat{e}_1\hat{k}\gamma_y}{2c(p_2k)}\right)\Lambda_1'\\&\quad+(g^2-1)\frac{e^2A_0^2(ke_1)\hat{k}}{2c^2(p_1k)(p_2k)}\Lambda_2\Bigg\}u(p_1),\end{aligned} \tag{2.12}$$

where we have introduced the following functions [10]:

$$\begin{aligned}&\{\sin\varphi,\cos^n\varphi\}\exp\{i[\alpha\sin(\varphi-\varphi_0)-\beta\sin2\varphi]\}\\&=\sum_s\{\Lambda_1'(\alpha,\beta,s),\Lambda_n(\alpha,\beta,s)\}\exp(is\varphi)\end{aligned} \tag{2.13}$$

and the parameters are defined as follows:

$$\begin{aligned}\alpha=&\frac{eA_0}{\hbar c}\Bigg\{\left(\frac{p_{1x}}{(p_1k)}-\frac{p_{2x}}{(p_2k)}\right)^2\\&+g^2\left(\frac{p_{1y}}{(p_1k)}-\frac{p_{2y}}{(p_2k)}\right)^2\Bigg\}^{1/2},\end{aligned} \tag{2.14}$$

$$\beta=(g^2-1)\frac{e^2A_0^2}{8c^2\hbar}\left(\frac{1}{(p_1k)}-\frac{1}{(p_2k)}\right), \tag{2.15}$$

$$\sin\varphi_0=\frac{eA_0}{\alpha\hbar c}g\left(\frac{p_{1y}}{(p_1k)}-\frac{p_{2y}}{(p_2k)}\right), \tag{2.16}$$

and

$$\Delta(\mathbf{q}_1-\hbar\mathbf{k}'+s\hbar\mathbf{k},\mathbf{q}_1)=\frac{\varepsilon(\mathbf{q}_1-\hbar\mathbf{k}'+s\hbar\mathbf{k})-\varepsilon(\mathbf{q}_1)+\hbar\omega'}{\hbar} \tag{2.17}$$

is the resonance detuning.

We will use the Heisenberg representation, where the operator evolutions are given by the following equation:

$$i\hbar\frac{\partial\hat{L}}{\partial t}=[\hat{L},\hat{H}], \tag{2.18}$$

and the expectation values are determined by the initial density matrix $\hat{D}$

$$\langle\hat{L}\rangle=\mathrm{Sp}(\hat{D}\hat{L}). \tag{2.19}$$

Equation (2.18) should be supplemented by the Maxwell equation for $\overline{A}_e$ which is reduced to

$$\frac{\partial A_e}{\partial t}+\frac{c^2\mathbf{k}'}{\omega'}\frac{\partial A_e}{\partial\mathbf{r}}=-i\frac{4\pi c}{\omega'}\overline{\langle\hat{j}e_1\rangle\exp(ik'x)}, \tag{2.20}$$

where the overbar denotes averaging over time and space much larger than $(1/\omega',1/k')$ and

$$\langle\hat{j}e_1\rangle=\mathrm{Sp}(e_1\hat{j}\hat{D}), \tag{2.21}$$

$$\begin{aligned}e_1\hat{j}=&e\sum_s\sum_{\mathbf{q}_1,\sigma_1}\sum_{\mathbf{q}_2,\sigma_2}\hat{a}^{\dagger}_{\mathbf{q}_2,\sigma_2}\hat{a}_{\mathbf{q}_1,\sigma_1}\langle\mathbf{q}_2,\sigma_2\|s\|\mathbf{q}_1,\sigma_1\rangle\\&\times\exp\left[\frac{i}{\hbar}\{(\mathbf{q}_1-\mathbf{q}_2+s\hbar\mathbf{k})\mathbf{r}+(\varepsilon(\mathbf{q}_2)-\varepsilon(\mathbf{q}_1))t\}\right].\end{aligned} \tag{2.22}$$

As we are interested in amplification of the wave with a certain $\omega',\mathbf{k}'$ we can keep only resonant terms in Eq. (2.22) with $\mathbf{q}_2=\mathbf{q}_1-\hbar\mathbf{k}'+s\hbar\mathbf{k}$. In principle, because of the electron beam energy and angular spread different harmonics may contribute to the process considered, but in the quantum regime [see Eqs. (2.43) and (2.44)] we can keep only one harmonic $s=s_0$. For the resonant current amplitude we have the following expression:

$$-i\overline{(e_1\hat{j})\exp(-ik'x)}=\sum_{\mathbf{q}}\hat{\Pi}(\mathbf{q}), \qquad (2.23)$$

where

$$\hat{\Pi}(\mathbf{q})=-ie\sum_{\sigma_1,\sigma_2}\hat{a}^{\dagger}_{\mathbf{q}_-,\sigma_2}\hat{a}_{\mathbf{q},\sigma_1}\times\langle\mathbf{q}_-,\sigma_2\|s_0\|\mathbf{q},\sigma_1\rangle^{i\Delta(\mathbf{q}_-,\mathbf{q})t}. \qquad (2.24)$$

Here we have introduced the notation

$$\mathbf{q}_-=\mathbf{q}-\hbar\mathbf{k}'+s_0\hbar\mathbf{k}. \qquad (2.25)$$

The physical meaning of Eq. (2.25) is obvious: It describes the process where a particle with quasimomentum $\mathbf{q}$ is annihilated and a particle is created in the state with quasimomentum $\mathbf{q}-\hbar\mathbf{k}'+s_0\hbar\mathbf{k}$. Taking into account Eqs. (2.11), (2.18), and (2.10) for the operator $\hat{\Pi}(\mathbf{q})$, we obtain

$$\frac{\partial\hat{\Pi}(\mathbf{q})}{\partial t}-i\Delta(\mathbf{q}_-,\mathbf{q}_1)\hat{\Pi}(\mathbf{q})=\frac{e^2}{2\hbar c}\overline{A_e}\sum_{\sigma_1,\sigma_2,\sigma_3}\{\langle\mathbf{q},\sigma_1\|-s_0\|\mathbf{q}_-,\sigma_3\rangle\times\langle\mathbf{q}_-,\sigma_2\|s_0\|\mathbf{q},\sigma_1\rangle\hat{a}^{\dagger}_{\mathbf{q}_-,\sigma_2}\hat{a}_{\mathbf{q}_-,\sigma_3}-\langle\mathbf{q},\sigma_3\|-s_0\|\mathbf{q},_-,\sigma_2\rangle\langle\mathbf{q}_-,\sigma_2\|s_0\|\mathbf{q},\sigma_1\rangle\times\hat{a}^{\dagger}_{\mathbf{q},\sigma_3}\hat{a}_{\mathbf{q},\sigma_1}\}, \qquad (2.26)$$

where we have kept only resonant terms. These terms are predominant in near-resonant emission/absorption, since their detuning is much smaller than that of nonresonant terms, which are detuned from resonance by $\omega\gg|\Delta(\mathbf{q}_-,\mathbf{q})|$.

We will assume that the electron beam is nonpolarized. This means that the initial one-particle density matrix in momentum space is

$$\rho_{\sigma_1\sigma_2}(\mathbf{q}_1,\mathbf{q}_2,0)=\langle\hat{a}^{\dagger}_{\mathbf{q}_2,\sigma_2}(0)\hat{a}_{\mathbf{q}_1,\sigma_1}(0)\rangle=\rho_0(\mathbf{q}_1,\mathbf{q}_2)\delta_{\sigma_1,\sigma_2}. \qquad (2.27)$$

Here $\rho_0(\mathbf{q},\mathbf{q})$ is connected to the classical momentum distribution function $n(\mathbf{q})$ by the formula

$$\rho_0(\mathbf{q},\mathbf{q})=\frac{(2\pi\hbar)^3}{2}n(\mathbf{q}). \qquad (2.28)$$

For the expectation value of $\hat{\Pi}(\mathbf{q})$ from Eq. (2.26) we have

$$\frac{\partial\Pi(\mathbf{q})}{\partial t}-i\Delta(\mathbf{q}_-,\mathbf{q}_1)\Pi(\mathbf{q})=\frac{e^2M^2}{2\hbar c}\overline{A_e}[\rho(\mathbf{q}_-,\mathbf{q}_-,t)-\rho(\mathbf{q},\mathbf{q},t)], \qquad (2.29)$$

where

$$\rho(\mathbf{q}_1,\mathbf{q}_1,t)=\langle\hat{a}_{\mathbf{q}_1,\sigma_1}(t)\hat{a}_{\mathbf{q}_1,\sigma_1}(t)\rangle,$$

$$M^2=\sum_{\sigma_1,\sigma_2}\langle\mathbf{q},\sigma_1\|-s_0\|\mathbf{q}_-,\sigma_2\rangle\langle\mathbf{q}_-,\sigma_2\|s_0\|\mathbf{q},\sigma_1\rangle.$$

The M^2 is reduced to the usual calculation of a trace [12,10], and in our notation we have

$$M^2=\frac{2c^4}{q_{-0}q_0}\left|(pe_1')\Lambda_0+\frac{eA_0}{c}(e_{1x}'\Lambda_1+ge_{1y}'\Lambda_1')\right|^2, \qquad (2.30)$$

where

$$e_1'=e_1-k'\left(\frac{ke_1}{kk'}\right) \qquad (2.31)$$

Here we have neglected terms of the order of $(\hbar\omega'/q_0)^2\ll1$ as for a FEL this condition is always satisfied. Taking into account Eqs. (2.11), (2.18), and (2.10) for $\rho(\mathbf{q},\mathbf{q},t)$ and $\rho(\mathbf{q}_-,\mathbf{q}_-,t)$, we obtain

$$\frac{\partial\rho(\mathbf{q},\mathbf{q},t)}{\partial t}=\frac{1}{4\hbar c}(A_e^*\Pi+A_e\Pi^*), \qquad (2.32)$$

$$\frac{\partial\rho(\mathbf{q}_-,\mathbf{q}_-,t)}{\partial t}=-\frac{1}{4\hbar c}(A_e^*\Pi+A_e\Pi^*). \qquad (2.33)$$

To take into account pulse propagation effects we can replace the time derivatives by the following expression:

$$\frac{\partial}{\partial t}\rightarrow\frac{\partial}{\partial t}+\overline{\mathbf{v}}\frac{\partial}{\partial\mathbf{r}},$$

where $\overline{\mathbf{v}}$ is the mean velocity of the electron beam and the convectional part of the derivative expresses the pulse propagation effects. Introducing the new variables

$$\Delta n=\frac{(2\pi\hbar)^3}{2}[\rho(\mathbf{q}_-,\mathbf{q}_-,t)-\rho(\mathbf{q},\mathbf{q},t)], \qquad (2.34)$$

$$\frac{1}{(2\pi\hbar)^3}\Pi(\mathbf{q})=J(\mathbf{q}) \qquad (2.35)$$

and replacing the summation in Eq. (2.20) by integration, the self-consistent set of equations reads

$$\frac{\partial J(\mathbf{q})}{\partial t}+\overline{\mathbf{v}}\frac{\partial J(\mathbf{q})}{\partial \mathbf{r}}-i\Delta J(\mathbf{q})=\frac{e^2M^2}{4\hbar c}A_e(x,z,t)\Delta n(\mathbf{q}),$$

$$\frac{\partial \Delta n(\mathbf{q})}{\partial t}+\overline{\mathbf{v}}\frac{\partial \Delta n(\mathbf{q})}{\partial \mathbf{r}}=-\frac{1}{\hbar c}[A_e^*J(\mathbf{q})+A_eJ^*(\mathbf{q})],$$

$$\frac{\partial A_e}{\partial t}+\frac{c^2\mathbf{k}'}{\omega'}\frac{\partial A_e}{\partial \mathbf{r}}=\frac{4\pi c}{\omega'}\int d\mathbf{q}J(\mathbf{q}). \tag{2.36}$$

These equations yield the conservation laws for the energy of the system and particle number:

$$\frac{\partial|A_e|^2}{\partial t}+\frac{c^2\mathbf{k}'}{\omega'}\frac{\partial|A_e|^2}{\partial \mathbf{r}}=-\frac{4\pi\hbar c^2}{\omega'}\int d\mathbf{q}\left(\frac{\partial}{\partial t}+\overline{\mathbf{v}}\frac{\partial}{\partial \mathbf{r}}\right)\Delta n(\mathbf{q}),$$

$$\left(\frac{\partial}{\partial t}+\overline{\mathbf{v}}\frac{\partial}{\partial \mathbf{r}}\right)\left(\Delta n(\mathbf{q})^2+\frac{8}{e^2M^2}\left|J(\mathbf{q})\right|^2\right)=0. \tag{2.37}$$

Note that from the set of equations (2.36) one can obtain a small signal gain passing into perturbation theory which in the quasiclassical limit will coincide with the classical one.

In the quantum regime the emission and absorption are characterized by the widths

$$\Delta_e=s\omega\left(1-\frac{v}{c}\cos\theta_1\right)-\omega'\left(1-\frac{v}{c}\cos\theta\right)-\frac{e^2A_0^2\omega'}{2\varepsilon^2[1-(v/c)\cos\theta_1]}(1-\cos\theta_0)-\frac{s\hbar\omega\omega'}{\varepsilon}(1-\cos\theta_0), \tag{2.38}$$

$$\Delta_a=\Delta_e+\frac{2s\hbar\omega\omega'}{\varepsilon}(1-\cos\theta_0), \tag{2.39}$$

where θ_1 and θ are the incident and scattering angles of the pump and probe photons with respect to the electron beam direction of motion, and θ_0 is the angle of the pump and probe photons.

The quantum regime assumes that

$$\Delta_a-\Delta_e=\frac{2s\hbar\omega\omega'}{\varepsilon}(1-\cos\theta_0)>\max\left\{\left|\frac{\partial\Delta_e}{\partial\eta_i}\delta\eta_i+\frac{\partial^2\Delta_e}{\partial\eta_i^2}(\delta\eta_i)^2\right|,\frac{\omega}{N}\right\}, \tag{2.40}$$

where by η_i we denote the set of quantities characterizing the electron beam and pump field and by $\delta\eta_i$ their spreads. The second term in the curly brackets of Eq. (2.40) expresses the resonance width caused by the finite interaction length, and N is the number of periods of the pump field. In particular for the energetic $(\Delta\varepsilon)$ and angular $(\Delta\vartheta)$ spreads from Eq. (2.40) (for $\theta_0=\theta_1\simeq\pi,\ \theta\ll 1$) we will have

$$\Delta\varepsilon<\hbar\omega', \tag{2.41}$$

$$\left|\theta\Delta\vartheta+\frac{\Delta\vartheta^2}{2}\right|<\frac{4s_0\hbar\omega}{\varepsilon}. \tag{2.42}$$

The conditions for keeping only one harmonic $s=s_0$ in the resonant current are

$$\frac{\Delta\varepsilon}{\varepsilon}\ll 1/s_0, \tag{2.43}$$

$$\left|\theta\Delta\vartheta+\frac{\Delta\vartheta^2}{2}\right|<\frac{\omega}{\omega'}. \tag{2.44}$$

As we see, for not very high harmonics the conditions (2.43) and (2.44) are weaker than the conditions in the quantum regime Eqs. (2.41), (2.42), or (1.2) and are well enough satisfied for current accelerator beams.

III. STEADY-STATE REGIMES OF AMPLIFICATION

Our goal is to determine the conditions under which we will have nonlinear amplification. We assume steady-state operation, i.e., dropping of all partial time derivatives in Eqs. (2.36). The considered setup is either a single-pass amplifier for which an injected input signal is necessary, or self-amplified coherent spontaneous emission, for which a modulated beam is necessary. We will also consider the case of exact resonance neglecting detuning in Eqs. (2.36), assuming that the electron beam momentum distribution is centered at $\Delta_e=0$. To achieve maximal Doppler shift and optimal conditions of amplification we will assume counterpropagating electron and pump photon beams (Z axis, $\theta_0=\theta_1=\pi$). In this case the optimal condition for a LW pump is $\theta=0$, while for a CW pump $\theta\sim\xi/\gamma_L$ $(\theta\ll 1)$. For both cases we will assume that the envelope of the probe wave depends only on z. Then the set of equations (2.36) and conservation laws (2.37) are reduced to

$$\frac{\partial J}{\partial z}=\frac{e^2M^2}{4\hbar c\overline{v}_z}A_e\Delta n,$$

$$\frac{\partial \Delta n}{\partial z}=-\frac{2}{\hbar c\overline{v}_z}A_eJ,$$

$$\frac{\partial A_e}{\partial z}=\frac{4\pi}{\omega'}J, \tag{3.1}$$

$$\Delta^2+\frac{8}{e^2M^2}|\Pi|^2=N_0^2,$$

$$W = W_0 + \frac{\hbar\omega' \overline{v_z}}{2}(\Delta n_0 - \Delta n),$$

where N_0 is the beam density, W is the probe wave intensity, and W_0 is the initial probe wave intensity. From Eq. (3.1) we have the following expressions for J and Δn:

$$\Delta n = N_0 \cos\left\{ \frac{e|M|}{2^{1/2}\hbar c \overline{v_z}} \int_0^z A_e dz + \varphi_o \right\},$$

$$J = \frac{e|M|}{2^{3/2}} N_0 \sin\left\{ \frac{e|M|}{2^{1/2}\hbar c \overline{v_z}} \int_0^z A_e dz + \varphi_o \right\}, \tag{3.2}$$

where φ_o is determined by the boundary conditions. Denoting

$$\varphi = \frac{e|M|}{2^{1/2}\hbar c \overline{v_z}} \int_0^z A_e dz + \varphi_o \tag{3.3}$$

we arrive at the nonlinear pendulum equation

$$\frac{d^2\varphi}{dz^2} = \sigma^2 \sin\varphi, \tag{3.4}$$

where

$$\sigma^2 = \frac{\pi e^2 M^2 N_0}{\hbar\omega' c \overline{v_z}} \tag{3.5}$$

is the main characteristic parameter of amplification: $L_c = 1/\sigma$ is the characteristic length of amplification. For the LW from Eqs. (2.13), (2.14), (2.15), (2.16), and (2.30) we have

$$\sigma_L = \frac{\xi \Lambda_1(0,\beta,s_0)}{2\gamma_L^2} \sqrt{\alpha_0 \frac{c\lambda}{s_0 \overline{v_z}} N_0 (1+\xi^2/2)}. \tag{3.6}$$

Here λ is the wavelength of the pump wave, α_0 is the fine structure constant, and the function $\Lambda_1(0,\beta,s)$ is expressed by the ordinary Bessel functions:

$$\Lambda_1(0,\beta,s_0) \simeq \frac{1}{2}\left\{ J_{(s_0-1)/2}\left(\frac{s_0\xi^2}{4+2\xi^2}\right) - J_{(s_0+1)/2}\left(\frac{s_0\xi^2}{4+2\xi^2}\right)\right\}. \tag{3.7}$$

In this case only odd harmonics are possible. For the cw we have

$$\sigma_L = \frac{\xi}{2\gamma_L^2}\left(\frac{\theta\gamma_L}{\xi} + \frac{s_0}{\alpha}\right) J_{s_0}(\alpha) \times \sqrt{\alpha_0 \frac{c\lambda}{s_0 \overline{v_z}} N_0 (1+\xi^2+\theta^2\gamma_L^2)} \tag{3.8}$$

and the argument of the Bessel function is

$$\alpha \simeq \frac{2 s_0 \xi \theta \gamma_L}{1+\xi^2+\theta^2\gamma_L^2}. \tag{3.9}$$

We will consider two regimes of amplification that are determined by the initial conditions. For the first regime the initial macroscopic transition current of the electron beam is zero and it is necessary to have a seeding electromagnetic wave. In this case the following boundary conditions are imposed:

$$\Delta n|_{z=0} = N_0, \quad J|_{z=0} = 0, \quad W|_{z=0} = W_0. \tag{3.10}$$

The solution of Eq. (2.35) in this case reads

$$W(z) = W_0 \, dn^{-2}\left(\frac{\sigma}{\kappa} z; \kappa\right), \tag{3.11}$$

$$\kappa = \left(1 + \frac{W_0}{N_0 \hbar\omega' v_z}\right)^{-1/2}, \tag{3.12}$$

where $dn(z,\kappa)$ is the elliptic function of Jacobi and κ is its modulus.

As is known $dn(z,\kappa)$ is a periodic function with the period $2K(\kappa)$, where $K(\kappa)$ is the complete elliptic integral of first order. At the distances $L = (2r+1)\kappa K(\kappa)/\sigma (r = 0,1,2,\ldots)$ the wave intensity reaches its maximal value, which equals

$$W_{\max} = W_0 + N_0 \hbar\omega' v_z. \tag{3.13}$$

For the short interaction length $z \ll L_c$ from Eq. (2.35) we have

$$W(z) = W_0(1+\sigma^2 z^2)$$

and the wave gain is rather small. To extract maximal energy from the electron beam the interaction length should be at least of the order of half the spatial period of the wave envelope variation $\kappa K(\kappa)/\sigma$. At this condition the intensity value $W_{\max} = W_0 + N\hbar\omega' v_z$ is achieved, because all electrons make a contribution to the radiation field. Taking into account that the seed power is much smaller than $W_{\max}$ and that when $1-\kappa \ll 1$

$$K(\kappa) \rightarrow \frac{1}{2}\ln\left[\frac{16}{1-\kappa^2}\right],$$

we have for the amplification length

$$L \simeq L_c \ln\left(4\frac{W_{\max}}{W_0}\right). \tag{3.14}$$

Let us now consider the other regime of wave amplification when the electron beam is modulated and the ‘‘macroscopic transition current’’ J differs from zero. This regime can operate without any initial seeding power $(W_0 = 0)$. So we will consider the optimal case with the following initial conditions:

TABLE I. Characteristic parameters for different setups of electron beam and linearly polarized pump wave (lw).

s	γ_L	ξ	I(kA)	λ(cm)	$\hbar\omega'$ (eV)	$\hbar\omega'/\varepsilon$	L_c(cm)	L_s/L_c	$W_{\max}$ (W/cm^2)
1	16	5×10^{-3}	0.1	5×10^{-5}	2539	3.1×10^{-4}	4.3	65	7×10^{10}
1	30	10^{-2}	1	5×10^{-5}	8925	5.8×10^{-4}	2.36	29	2.4×10^{12}
1	40	10^{-2}	1	5×10^{-5}	15 866	7.76×10^{-4}	4.2	21	4×10^{12}
1	120	3×10^{-2}	1	10.6×10^{-4}	6733	10^{-4}	2.7	59	1.8×10^{12}
3	5	1	5	5×10^{-5}	496	2×10^{-4}	1.5×10^{-3}	9	1.3×10^{11}
11	2	1.5	5	5×10^{-5}	205	2×10^{-4}	1.5×10^{-3}	10	5.6×10^{10}
31	15	2.5	5	10.6×10^{-4}	791	10^{-4}	1.7×10^{-2}	10	10^{12}
51	20	3.5	5	10.6×10^{-4}	1340	1.3×10^{-4}	1.3×10^{-2}	6.2	1.8×10^{12}
61	10	4.5	5	5×10^{-5}	5437	10^{-3}	6.3×10^{-3}	0.3	7×10^{12}
101	40	4.5	5	10.6×10^{-4}	6795	3.3×10^{-4}	6×10^{-2}	1	9.2×10^{12}

$$J|_{z=0}=J_0, \qquad \Delta n|_{z=0}=\Delta n_0, \qquad W|_{z=0}=0. \quad (3.15)$$

Then the wave intensity is expressed by the formula

$$W(z)=\frac{N_0\hbar\omega' v_z}{2}\left(1-\frac{\Delta n_0}{N_0}\right)\left[\frac{1}{dn^2(\sigma z;k)}-1\right] \quad (3.16)$$

and the modulus is

$$\kappa^2=\frac{1}{2}\left(1+\frac{\Delta n_0}{N_0}\right). \quad (3.17)$$

As is seen from Eq. (3.16) in this case the intensity varies periodically with the distance as well, with the maximal value of the intensity

$$W'_{\max}=\frac{N_0\hbar\omega' v_z}{2}\left(1+\frac{\Delta n_0}{N_0}\right). \quad (3.18)$$

The second regime is more interesting. It is the regime of generation without initial seeding power and has a superradiant nature. For a short interaction length $z \ll L_c$ according to Eq. (3.16)

$$W(z)=\frac{N_0\hbar\omega' v_z \sigma^2 z^2}{4}\left(1-\frac{\Delta n_0}{N_0}\right). \quad (3.19)$$

The intensity scales as N_0^2 ($\sigma^2 \sim N_0$) which means that we have superradiation. The radiation intensity in this regime reaches a significant value even at $z \ll L_c$.

IV. DISCUSSION

The coherent interaction time of electrons with probe radiation is confined by several relaxation processes. To be more precise, in the self-consistent set of equations (2.36) we should add terms describing spontaneous transitions and other relaxation processes. Since we have not taken into account the relaxation processes, this consideration is correct only for distances $L < c\tau_{\min}$, where $\tau_{\min}$ is the minimum of all relaxation times. Due to spontaneous radiation electrons will lose energy $\sim\hbar\omega'$ at distances

$$L_s \simeq c\frac{\hbar\omega'}{W_s}=\frac{3}{2\pi}\frac{s_0\lambda}{\alpha_0(1+\xi^2/2)\xi^2},$$

where W_s is the power of the spontaneous radiation (for a lw; for a cw one should replace $\xi^2 \to 2\ \xi^2$). Although the cutoff harmonic increases with increasing ξ ($s_c \sim \xi^3$), for high laser intensities $\xi > 1$ the role of spontaneous radiation becomes essential since $L_s \sim \xi^{-4}$ and the above mentioned regimes will be interrupted. Therefore the solutions obtained are correct at distances $L \leqslant L_s$.

In Tables I and II we give the parameters for the different setups of beam and pump fields for lw's as well as for cw's. The beam radius has been chosen as 10^{-3} cm. By I we denote the beam current. As we see from these tables for high harmonics L_c decreases and simultaneously the quantum recoil $\hbar\omega'/\varepsilon$ increases, but $L_s \sim L_c$. The first regime will effectively work as a single-pass amplifier if $L_s \gtrsim 10\ L_c$. In this regime, it follows from Eq. (3.14) that

TABLE II. Characteristic parameters for different setups of electron beam and circularly polarized pump wave (cw).

s	γ_L	ξ	I(kA)	λ(cm)	$\hbar\omega'$ (eV)	$\hbar\omega'/\varepsilon$	L_c(cm)	L_s/L_c	$W_{\max}$ (W/cm^2)
3	5	0.7	5	5×10^{-5}	375	1.5×10^{-4}	1.4×10^{-3}	7.3	10^{11}
50	20	2	5	10.6×10^{-4}	1039	10^{-4}	2.2×10^{-2}	4.6	1.4×10^{12}
110	25	3	5	10.6×10^{-4}	1693	1.3×10^{-4}	1.7×10^{-2}	2.68	2.3×10^{12}
220	20	3	5	10.6×10^{-4}	2166	2.1×10^{-4}	7.8×10^{-2}	1.2	3×10^{12}
400	8	3.5	5	10.6×10^{-4}	470	1.1×10^{-4}	2.4×10^{-2}	3.87	6.4×10^{11}

$$W_{\max} \simeq e^{L/L_c} W_0/4.$$

As we see from Tables I and II the condition $L_s/L_c \gtrsim 10$ is satisfied for the fundamental frequency as well as for the harmonics.

Concerning the conditions when the quantum regime under consideration will operate, we can see from Tables I and II that for average energies of the electron beam $\varepsilon \sim 10\text{–}100$ MEV we have $\hbar\omega'/\varepsilon \sim 5\times 10^{-4}$ and the conditions (2.41) and (2.42) demand the following energy spread and emittance for the electron beam:

$$\frac{\Delta\varepsilon}{\varepsilon} < 5\times 10^{-4}, \qquad \Delta\vartheta < \gamma^{-1}\sqrt{\frac{\hbar\omega'}{\varepsilon}} \sim 10^{-3},$$

which are actually attainable in modern accelerators.

Finally, note that the above mentioned second regime may be more promising as it allows considerable output intensities even for small interaction lengths [Eq. (3.19)]. It is expected that the effects of energy and angular spread will not be significant in this regime as it is governed by the initial current and only Doppler dephasing and spontaneous lifetime will interrupt the superradiation process. This problem, as well as the influence of spontaneous radiation on the generation process, will be the subject of further investigation. Note only that the initial quantum modulation of the particle beams at the above optical frequencies necessary for the second regime can be obtained through multiphoton transitions in the laser field in the presence of a "third body." The possibilities of quantum modulation at hard x-ray frequencies in induced Compton, undulator, and Cherenkov processes were studied in [13].

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